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The Two-Criterial Dynamic Lot Size Problem

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The dynamic lot size model with time-constant costs is studied. The criterion of minimizing the sum of set-up and holding costs is complemented by another objective consisting in the minimization of the stock. It is shown that solutions which are efficient with respect to these two objectives can be derived from parametric one-criterial models with combined objective functions. A complete set of efficient solutions which are distinct in their objectives is covered by this approach.

1. The problem

Most of the textbooks in Operations Research touch in one or another way lot size models or their dynamic extensions [2], [3], [7], [9]–[11]. Generalizations of the models in multi-stage processes and multi-item systems have been also investigated [1], [4]–[7], [12]. Models of this type are widely used in practice to determine the size of production by minimizing the sum of size-independent fixed costs and size-depending holding costs. It follows usually from the application of such models that the size of production lots will raise, and, that the period of production will be moved away from the period in which the demand actually must be satisfied. The greater the distance between the periods of producing and selling the items, the more probable are changes in the demand structure and the more complicated will it be to respond flexibely to disturbances.

It was this situation sketched here that suggested to study multi-criterial lot size models. The economical criterion of minimizing the costs can be complemented by another one, which might be treated as minimizing the difference between the periods of producing and selling the items. More concretely, it finds its expression in the minimization of the stock. Then a two-criterial lot size model can be formulated and solutions which are efficient with respect to the costs and to the stock are to be found. The decision maker will then choose that solution which suits best his individual conception on the relationsship between costs and time.

In the paper the model will be described and it will be shown that the efficient solutions can be found by applying the stability results for the one-criterial problem [8] to the model with combined objective functions.

The one-criterial lot size model can be introduced as follows: The process of production and stock-holding is considered for one item and T periods. The production figures $x_t \geq 0$ for t^{th} period, $t = 1, 2, \ldots, T$, have to be chosen such that the given deterministic demand $d_t \geq 0$ is satisfied for all t and that the total sum of set-up costs and holding costs is minimal. The fixed set-up costs arising if $x_t > 0$ are denoted by $c > 0$ and the per-unit per-period linear holding costs are denoted by $h > 0$. If the stock at the end of t^{th} period is denoted by y_t the demand is satisfied in the case that $y_t = y_{t-1} + x_t - d_t$ is nonnegative. Usually the assumption is made that the stock equals zero at the begin and at the end of the considered planning period $1, 2, \ldots, T$, i.e. $y_0 = y_T = 0$. Then the model can be described by the formulae (1)–(4):

$$y_t = y_{t-1} + x_t - d_t, \qquad t = 1, 2, \ldots, T, \tag{1}$$

$$y_t \geq 0, \quad x_t \geq 0, \tag{2}$$

$$y_0 = y_T = 0, \tag{3}$$

$$F_1 = c \sum_{t=1}^{T} \operatorname{sign} x_t + h \sum_{t=1}^{T} y_t \to \min. \tag{4}$$

The second criterion consisting in the minimization of the stock is provided by formula (5):

$$F_2 = \sum_{t=1}^{T} y_t \to \min. \tag{5}$$

Note that F_2 makes sense only if the first criterion is also of interest. In the other case the solution $\{x_t = d_t\}_{t=1}^{T}$ provides $F_2 = 0$.

The one-criterial model will play a significant role in the subsequent investigations. It will be therefore denoted by $M(c, h)$. Feasible solutions of this model have to satisfy the conditions (1)–(3), while optimal solutions are feasible solutions minimizing the function (4).

Let two feasible solutions be considered which have the values F_1', F_2' and F_1'', F_2'' respectively. Then the first solution is said to be dominated by the second one if $F_1' \geq F_1''$ and $F_2' \geq F_2''$. It is strongly dominated if the inequalities hold and one of them is strong. An efficient solution can be defined as a feasible solution which is not strongly dominated by any other feasible solution. In other words, the feasible solution associated with the values F_1', F_2' is efficient, if it follows for any other feasible solution that $F_1'' < F_1'$ implies $F_2' < F_2''$ and $F_2'' < F_2'$ implies $F_1' < F_1''$.

The main task in multi-criterial optimization is to find set EFF of all efficient solutions of a given problem. In many cases it is preverable to characterize subset EFF' of efficient solutions which covers all possible pairs of values (F_1, F_2) over EFF, i.e. EFF' contains at least one efficient solution for each pair of values. The aim of the paper is to describe such subset for model (1)–(5).

Example: Let $T = 3$, $c = 5$, $h = 2$, $d_1 = 3$, $d_2 = 2$, $d_3 = 1$. The optimal solution $(x, y) = \{x_1 = 3, x_2 = 3, x_3 = 0, y_1 = 0, y_2 = 1, y_3 = 0\}$ with $F_1 = 12$ and $F_2 = 1$ is an efficient solution as it will be shown later on.

2. Stability and monotony of the one-critical problem M (c, h)

In this section some results from our paper [8] will be listed in a slightly modified version and new monotony properties will be derived.

It can be easily shown that all optimal solutions of $M(c, h)$ fulfil

(i) $d_t = 0$ implies $x_t = 0$,
(ii) $y_t > 0$ implies $x_t = 0$ and
(iii) $y_t = 0$ implies $x_t \in \left\{d_t, d_t + d_{t+1}, \ldots, \sum_{r=t}^{T} d_r\right\}$ for all t.

The optimal solutions can be found using the following recursive procedure: Let $T_0 = \{t : d_t > 0\}$, $h(k, l) = h \sum_{t=k+1}^{l} (t - k - 1)\, d_t$ and $c(k, l) = c + h(k, l)$ for $k < 1$

and $d_{k+1} > 0$. Then

$$f_0 := 0, \quad f_t := \min \{c(k, t) + f_k : k \leq t - 1, k \in T_0 - 1\} \tag{6}$$

for all $t \geq 1$ and $t \in (T_0 - 1) \cup \{T\}$, can be used to determine the minimal value $F_1 = f_T$ for $M(c, h)$.

Let $f(k, t) = c(k, t) + f_k$ and let the parameters $k(t)$ be introduced by

$$f_t = f(k(t), t) \tag{7}$$

for all suitable t. Then an optimal solution can be found using these parameters.

Algorithm: Input $k(t)$ for $t \in (T_0 - 1) \cup \{T\}$.

1. $t := T$.
2. $x_{k(t)+1} := \sum_{r=k(t)+1}^{t} d_r, \quad x_r := 0 \quad$ for $\quad r = k(t) + 2, \ldots, t$.
3. If $k(t) = 0$ stop, else $t := k(t)$, go to step 2.

Output $x_1, x_2, \ldots, x_T$.

The components of the vector y can be determined by (1).

Since the parameters $k(t)$ play the most important role in solving the one-criterial problems the collection $K = \{k(t)\}_{t \in (T_0-1) \cup \{T\}}$ is called generalized solution. K can be used to solve not only $M(c, h)$ but a class of problems for all time horizons $T' \leq T$. The parameters $k(t)$ are usually not unique. It can be, however, shown, that for all generalized solutions the inequality

$$\max \{k(l)\} \leq \min \{k(t)\} \tag{8}$$

for all $l < t$ holds.

Example: The optimal solution provided in the first section is generated from $K = (0, 0, 1)$. The collection $K' = (0, 1, 1)$ leads to the same optimal solution, it is, however, not a generalized solution, for the optimal solution for $T' = 2$ cannot be derived from this collection. If the cost inputs are replaced by $(c, h) = (4, 2)$, then both $K = (0, 0, 1)$ and $K' = (0, 1, 1)$ are generalized solutions and max $k(2) = \max (0, 1) = k(3) = 1$.

It follows from the approach used to solve problem $M(c, h)$ that it is preverable to study the stability of the model in terms of generalized solutions and not in terms of optimal solutions.

Let K be a generalized solution and let the following parameters be introduced:

(i) $\bar{c}(0) := 0, \ \bar{c}(t) := \bar{c}(k(t)) + 1$ for all suitable t.

(ii) $r(k, t) = (f(k, t) - f_t)/(\bar{c}(k(t)) - \bar{c}(k))$ for all suitable k, t.

(iii) $l = \max\limits_t \max\limits_k \{r(k, t) : \bar{c}(k(t)) < \bar{c}(k)\}$ and

$u = \min\limits_t \min\limits_k \{r(k, t) : \bar{c}(k(t)) > \bar{c}(k)\}$,

where l and u can be set minus or plus infinite respectively, if they are not defined.

Theorem 1 ([8]): *K is a generalized solution for all $M(c', h')$ with $c', h' > 0$ and $(c + l)/h \leq c'/h' \leq (c + u)/h$ and it is not a generalized solution for any other pair (c', h').*

Theorem 2 ([8]): *Let $d_1, d_2, \ldots, d_T$ be a fixed sequence of demand values. Then there is a finite number of generalized solutions and associated stability regions defined by the inequalities in Theorem 1 which cover $\mathbb{R}^2_+$.*

Example: For the example from Section 1 the values $l = -1$ and $u = 1$ can be found, i.e. the stability region of $K = (0, 0, 1)$ is provided by $2 \leq c'/h' \leq 3$. For the demand vector (3, 2, 1) the positive orthant $\mathbb{R}^2_+$ is covered by the stability regions

$c' \leq h', h' \leq c' \leq 2h', 2h' \leq c' \leq 3h', 3h' \leq c'$ with the associated generalized solutions (0, 1, 2), (0, 1, 1), (0, 0, 1) and (0, 0, 0) (compare Fig. 1).

Now some monotony results will be derived.

Lemma 1: *Let K be a generalized solution for $M(c, h)$ and let $u = 0 = r(k'', t'')$. Then the collection* $K'' = \begin{cases} k(t), & t \neq t'' \\ k'', & t = t'' \end{cases}$ *is also a generalized solution for $M(c, h)$.*

Proof: It follows from $0 = r(k'', t'')$ that $f(k'', t'') = f_{t''}$. Then all values f_t remain unchanged and the new solution has the same value as K.

Remark: A similar result holds for $l = 0$.

Lemma 2: *Let $t' = \max \{t \in T_0 - 1 : t < t'', d_{t+1} > 0\}$. Then the inequalities $k'' \geq k(t')$ and $k(t'') \leq t'$ are fulfilled.*

Proof: It follows from inequality (8), Lemma 1 and relation $k(t'') < t''$ that the statement is valid.

Lemma 3: $\bar{c}(t') \geq (t'')\bar{c} - 1$.

Proof: Let $\bar{c}(t') \leq \bar{c}(t'') - 2$. It has been shown in [8] that $k < l$ implies $\bar{c}(k) \leq \bar{c}(l)$. If $k(t'') \leq t'$ then the inequalities $\bar{c}(t') \leq \bar{c}(k(t'')) - 1$ and $\bar{c}(k(t'')) \leq \bar{c}(t')$ hold. Contradiction. Hence $k(t'') > t'$ holds contradicting Lemma 2.

Theorem 3: *Let K be a generalized solution for $M(c, h)$ and let K'' be the generalized solution generated in Lemma 1 for $M(c + u, h)$. Let (x, y) and (x'', y'') be the corresponding optimal solutions. Then the inequality Σ sign $x_t \leq \Sigma$ sign $x_t'' + 1$ holds.*

Proof: The generalized solutions K and K'' differ only in t''. It follows from Lemmata 2/3 that $\bar{c}(k'') \geq \bar{c}(k(t')) \geq \bar{c}(k(t'')) - 1$, i.e. the number of periods of production will drop by at most one unit. Then the same will be true for $t = T$.

Theorem 4: *Let $d_1, d_2, \ldots, d_T$ be a fixed sequence of demand values. Then there are $|T_0|$ generalized solutions $K_1, K_2, \ldots, K_{|T_0|}$ with the corresponding optimal solutions $(x^1, y^1), (x^2, y^2), \ldots, (x^{|T_0|}, y^{|T_0|})$ in the cover of $\mathbb{R}^2_+$ such that*

$$\Sigma \text{ sign } x_t^s = |T_0| - s + 1, \qquad s = 1, 2, \ldots, |T_0|.$$

Proof: The collection $\{k - 1\}_{k \in T_0}$ is a generalized solution for sufficiently small c'. Then determining the parameters u and generating new solutions as in Lemma 1 the number of production periods will gradually drop by one unit or will remain unchanged. Since, on the other hand, $\{0\}_{k \in T_0}$ is a generalized solution for sufficiently large c', there are $|T_0|$ generalized solutions with the property indicated in the statement.

Remark: The solutions mentioned in the theorem are optimal with respect to an appropriate sequence of cost inputs $c_1/h_1 \leq \ldots \leq c_{|T_0|}h_{|T_0|}$.

Theorem 5: *Let two pairs of cost inputs $0 < c'/h' < c''/h''$ be given and let (x', y') and (x'', y'') be optimal solutions associated with the cost inputs. Then the following inequalities hold.*

(i) $\sum_{t=1}^T y_t' \leq \sum_{t=1}^T y_t''$ and

(ii) $\sum_{t=1}^T \text{sign } x_t' \geq \sum_{t=1}^T \text{sign } x_t''$.

Proof: (i) Let $\sum_{t=1}^T y_t' > \sum_{t=1}^T y_t''$. If (ii) holds then $c' \Sigma \text{ sign } x_t' + h' \Sigma y_t' > c' \Sigma \text{ sign } x_t'' + h' \Sigma y_t''$ is fulfilled. Contradiction. In the other case let $d = \Sigma \text{ sign } x_t' - \Sigma \text{ sign } x_t'' < 0$ and $g = \Sigma y_t' - \Sigma y_t'' > 0$. Then

$$c' \Sigma \text{ sign } x_t' + h' \Sigma y_t' - c' \Sigma \text{ sign } x_t'' - h' \Sigma y_t'' = c'd + h'g \leq 0 \quad \text{and}$$

$$c'' \Sigma \text{ sign } x_t' + h'' \Sigma y_t' - c'' \Sigma \text{ sign } x_t'' - h'' \Sigma y_t'' = c''d + h''g \geq 0.$$

It follows from these inequalities that $c'd \leq -h'g < -c'h''g/c'' = -h''gc'/c'' \leq c''dc'/c'' = c'd$. Contradiction. (ii) Let $\Sigma \operatorname{sign} x'_t < \Sigma \operatorname{sign} x''_t$. Then it follows from the first statement of the theorem that (x'', y'') is not optimal. Contradiction.

3. Complete characterization of *EFF'*

It will be shown in this section that the efficient solutions from EFF' can be found using the results provided in the previous chapter. One can easily prove that efficient solutions satisfy the relations (i)–(iii) for optimal solutions from Section 2. Therefore the set of feasible solutions of the two-criterial problem can be assumed finite and the following result will be of some help.

Theorem 6: *Let F_1 and F_2 be two objective functions given on a finite set X. Then all optimal solutions of the parametric problem*

$$aF_1(x) + (1 - a) F_2(x) \to \min \tag{9}$$

subject to $x \in X$ for $a \in (0, 1)$ and at least one optimal solution for $a = 0$ and $a = 1$ respectively are efficient with respect to $F_1 \to \min$, $F_2 \to \min$ and $x \in X$.

The parametric problem (9) for (1)–(5) is given by

$$M(ac, ah + (1 - a)). \tag{10}$$

Regarding all possible generalized solutions K for some pair $(ac, ah + (1 - a))$ the corresponding optimal solutions can be generated by the algorithm from the first section.

Theorem 7: *All generalized (and therefore also optimal) solutions of the problem* (10) *for $a < 1$ are efficient solutions for the model* (1)–(5).

Proof: If a is positive then the statement is a corollary of the previous theorem. Let $a = 0$. Then the objective function reduces to F_2 and there is only one optimal solution which is efficient.

Example: Let the example from the first section be considered. Then the generalized solutions associated with the cost inputs $(5a, 1 + a)$ for $0 \leq a < 1$ are efficient solu-

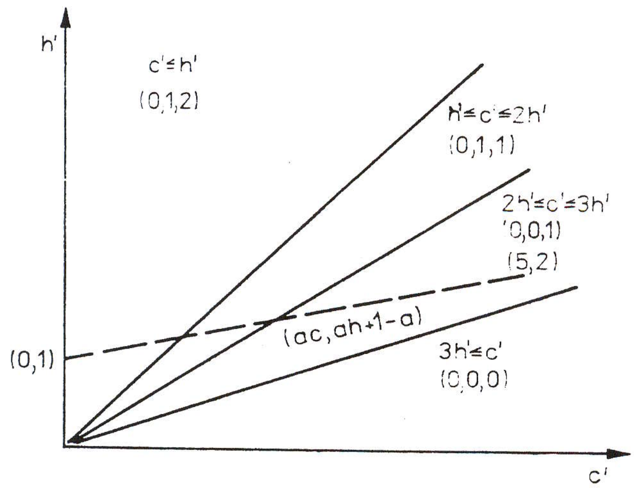

Fig. 1

tions. It can be seen from Fig. 1 that actually (0, 1, 2), (0, 1, 1) and (0, 0, 1) are the corresponding efficient solutions. Let a' be such real number that

(i) $0 < a' < 1$ and that

(ii) the corresponding generalized solution can be used to solve the problem (10) for $a = 1$.

It is clear that a' exists for, if $l < 0$, then a generalized solution for (c, h) is also solution for $a < 1$ and

$$(c + l)/h \leq ac/(ah + (1 - a))$$

(compare Fig. 1). If $l = 0$ then (c, h) lies on the boundary of the stability region and the generalized solution associated with the left neighbour region is also optimal for (c, h) and it can be generated from some $a < 1$.

Theorem 8: *The generalized solutions for $M(c', h')$, $c'/h' \geq c/h$, are dominated by the generalized solution for $M(a'c, a'h + (1 - a'))$.*

Proof: The values F_1 cannot be less than that of the generalized solution for $(a'c, a'h + (1 - a'))$ since the latter is an optimal solution with respect to (c, h). It follows from Theorem 5(i) that the same is true for F_2.

Example: It can be seen from Fig. 1 that (0, 0, 0) is the generalized solution for $3h' \leq c'$ and $c'/h' \geq 3 > 5/2 = c/h$. Then

$$F_1 = 12 < F_1' = c + 4h = 13 \quad \text{and} \quad F_2 = 1 < F_2' = 4.$$

It follows from the previous two theorems that efficient solutions can be derived from the generalized and optimal solutions for $M(\cdot, \cdot)$.

The question is now whether solutions which are not optimal for any pair (c', h') may be efficient.

Theorem 9: *Let K be a collection of integers which is not a generalized solution for any pair of cost inputs. Then it is dominated by one of the generalized solutions.*

Proof: According to Theorems 4/5 there are exactly $|T_0|$ generalized solutions $K_1, K_2, \ldots, K_{|T_0|}$ and corresponding optimal solutions associated with cost inputs (c_1, h_1), $c_2, h_2)$, $\ldots$, $(c_{|T_0|}, h_{|T_0|})$ such that $c_s/h_s \leq c_{s+1}/h_{s+1}$,

$$\Sigma \operatorname{sign} x_t^s = |T_0| - s + 1,$$

$$\Sigma y_t^s \leq \Sigma y_t^{s+1}, \quad s = 1, 2, \ldots, T_0(-1).$$

Let s' indicate the index of the initial cost inputs, i.e. $(c, h) = (c_{s'}, h_{s'})$, and let (x, y) be the feasible solution generated from collection K. Then there is some s'' such that $\Sigma \operatorname{sign} x_t = \Sigma \operatorname{sign} x_t^{s''}$.

(i) Let $s'' \geq s'$. Then $\Sigma y_t^{s''} \geq \Sigma y_t^{s'}$ and (x, y) is dominated by $(x_t^{s'}, y_t^{s'})$.

(ii) Let $s'' < s'$. Then $\Sigma y_t^{s''} \leq \Sigma y_t$ since otherwise $K_{s''}$ is not a generalized solution and K is dominated by $K_{s''}$.

It follows from the statements that the elements of EFF' can be derived from the generalized solutions for problem (10).

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